$$G^d_{n-1} \simeq \frac{biv}{2\pi}\varepsilon_{n-1}\sin^{-2}\beta\cos\left(\frac{v}{\sin\beta}z\right)Y^d\left(\frac{v}{\sin\beta}\right)\Big|_{r=0,\ l=n-1}$$

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ON THE RATE OF PROPAGATION OF SMALL PERTURBATIONS IN POROUS MEDIA*

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The properties of a system of equations which describes the two-speed motion of a porous medium are investigated. The type of system of equations is defined as a function of the rate of slippage of the phases and the difference in the stresses in the phases. The domains of variation of the decisive parameters for which the system of equations describing the dynamics of a two-phase porous medium remains hyperbolic are established.

For the correct formulation of the problem of the two-speed flow of a compressible porous medium it is necessary to determine the type of corresponding system of differntial equations. There are a considerable number of papers dealing with similar kinds of investigations for various systems of equations describing the motion of multiphase media. The equations of continuity and the equations of motion are written out for each phase: an assumption concerning barotropicity is used for the closure of the system and the non-hyperbolic nature of such a system of equations is indicated for real values of the difference in the speeds of the phases /1, 2/. It has been shown /3/ that, in the more general case for the complete system of equations which describes the flow of compressible phases using a model containing the same pressure for the different phases, the system of differential equations is not hyperbolic for real values of the magnitude of slippage. The propagation of small perturbations in a mixture with a barotropic gas phase has been investigated: it was noted that the non-hyperbolic character and instability of the small perturbations which are typical of the system of differential equations are attributable to an insufficiently complete description of the interphase interactions within the disperse

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phase /4/. When different pressures, which determined independently for each phase, are introduced, the system of equations becomes hyperbolic /5/. Inequalities are obtained which define the type of system in the space of the physical variables in the case of an incompressible solid phase and the overall pressure for the two phases*. (*Kazakov Yu.V., Fedorov A.V. and Fomin V.M., Investigation of the structures of isothermal shock waves and the calculation of the dispersion of a cloud of a gaseous suspension, Preprint 8, Inst. Teor. i Prikladn. Mekhaniki, Sib. Otd. Akad. Nauk SSSR, Novosibirsk, 1986.)

The system of equations describing the one-dimensional, non-stationary flow of a two-phase porous mixture when there is no interphase mass exchange has the form /6/

$$\frac{\partial\rho^\circ_i\alpha_i}{\partial t}+\frac{\partial\rho_i^\circ\alpha_iu_i}{\partial x}=0 \tag{1}$$
$$\frac{\partial\rho_i^\circ\alpha_iu_i}{\partial t}+\frac{\partial\rho_i^\circ\alpha_iu_i^2}{\partial x}+\frac{\partial\alpha_ip_i}{\partial x}-p_1\frac{\partial\alpha_i}{\partial x}=(-1)^iF$$
$$\frac{\partial\rho_i^\circ\alpha_i(^1/_2u_i^2+e_i)}{\partial t}+\frac{\partial\rho_i^\circ\alpha_iu_i(^1/_2u_i^2+e_i)}{\partial x}+\frac{\partial\alpha_ip_iu_i}{\partial x}-p_1u_2\frac{\partial\alpha_i}{\partial x}+p_1\frac{d_2\alpha_i}{dt}=$$
$$(-1)^i(Fu_2+Q);\quad \frac{d_i}{dt}=\frac{\partial}{\partial t}+u_i\frac{\partial}{\partial x},\qquad i=1,2$$

Here, ρ_i°, u_i, e_i and α_i are the true density, velocity, internal energy and mean bulk concentration of the i-th phase $(i=1,2)$, p_1 is the pressure in the gas, p_2 is the longitudinal stress in the condensed phase taken with the opposite sign and F and Q characterize the interphase friction and heat exchange.

We supplement the system with the equations of state for the gas $p_1=\rho_1^\circ RT_1$ and for the solid phase under the assumption that there is a planar deformed state locally:

$$\varepsilon_{rr}=ap_1-bp_2,\quad \varepsilon_{xx}=ap_2-bp_1 \tag{2}$$
$$a=(1-\nu)(1+\nu)/E,\quad b=\nu(1+\nu)/E,\quad \rho_i=\rho_i^\circ\alpha_i$$

Here, ε_{rr} is the strain in the transverse direction and ε_{xx} is the strain in a longitudinal direction and the deformations are positive upon compression. E and ν are Young's modulus and Poisson's ratio, respectively.

After some reduction the system of Eqs.(1) can be reduced to the form

$$\frac{\partial\rho_i^\circ\alpha_i}{\partial t}+\frac{\partial\rho_i^\circ\alpha_iu_i}{\partial x}=0,\quad i=1,2,\quad \frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}+\frac{1}{\rho_1^\circ}\frac{\partial p_1}{\partial x}=-\frac{F}{\rho_1} \tag{3}$$
$$\frac{\partial u_2}{\partial t}+u_2\frac{\partial u_2}{\partial x}+\frac{1}{\rho_2^\circ}\frac{\partial p_2}{\partial x}+\frac{p_1-p_2}{\rho_2}\frac{\partial\alpha_1}{\partial x}=\frac{F}{\rho_2}$$
$$\frac{\partial e_1}{\partial t}+u_1\frac{\partial e_1}{\partial x}+\frac{p_1}{\rho_1^\circ}\frac{\partial u_1}{\partial x}-\frac{p_1(u_2-u_1)}{\rho_1}\frac{\partial\alpha_1}{\partial x}+\frac{p_1}{\rho_1}\left(\frac{\partial\alpha_1}{\partial t}+u_2\frac{\partial\alpha_1}{\partial x}\right)= \tag{4}$$
$$-\frac{F(u_2-u_1)}{\rho_1}-\frac{Q}{\rho_1}$$
$$\frac{\partial e_2}{\partial t}+u_2\frac{\partial e_2}{\partial x}+\frac{p_2}{\rho_2^\circ}\frac{\partial u_2}{\partial x}-\frac{p_1}{\rho_2}\left(\frac{\partial\alpha_1}{\partial t}+u_2\frac{\partial\alpha_1}{\partial x}\right)=\frac{Q}{\rho_2} \tag{5}$$

In addition to this, we note that the following differential relationship exists between ε_{rr} and α_1:

$$\frac{\partial\alpha_1}{\partial t}+u_2\frac{\partial\alpha_1}{\partial x}-\alpha_2\left(\frac{\partial\varepsilon_{rr}}{\partial t}+u_2\frac{\partial\varepsilon_{rr}}{\partial x}\right)=0 \tag{6}$$

while the kinematic relationship between the strain in the longtudinal direction ε_{xx} and the velocity u_2 is expressed by the formula

$$\frac{\partial\varepsilon_{xx}}{\partial t}+u_2\frac{\partial\varepsilon_{xx}}{\partial x}+\frac{\partial u_2}{\partial x}=0 \tag{7}$$

Let us now find the characteristic directions in the x, t plane of the system of differential equations.

When account is taken of the equation of continuity of the gas phase and relationship (6), Eq.(4) reduces to the form

$$T_1\frac{d_1s_1}{dt}=-\frac{F(u_2-u_1)}{\rho_1}-\frac{Q}{\rho_1} \tag{8}$$

By taking account of Eq.(7), the energy equation for the solid phase can be written in the following form:

$$T_2 \frac{d_2 s_2}{dt} = \frac{Q}{\rho_2} \tag{9}$$

So, the equations for the internal energy of each phase yield two characteristic directions which are determined by the velocity fields of the first and second phases and which correspond to the transmission of thermal perturbations.

On integrating Eq.(6) and taking account of the inequalities (2), we obtain the dependence

$$\alpha_1 = f(p_1, p_2) \quad (f(p_1, p_2) = 1 - \alpha_{20} \exp(bp_2 - ap_1)) \tag{10}$$

When account is taken of the equation of continuity for the solid phase, Eq.(7) yields the integral

$$\rho_2^\circ = g(p_1, p_2) \quad (g(p_1, p_2) = \rho_{20}^\circ \exp((p_1 + p_2)(a - b))) \tag{11}$$

Hence, the remaining characteristic directions are the characteristic directions of system (3).

Let us now transform this system and write it in terms of the unknown functions p_1, p_2, u_1 and u_2. In order to do this, we will introduce the velocity of sound in the "pure" gas a_1 and make use of the equation of state of the gas $p_1 = \rho_1^\circ R T_1$.

In the new variables, system (3) will have the form

$$k_1 \frac{d_1 p_1}{dt} + \rho_1^\circ f_{,2} \frac{d_1 p_2}{dt} + \rho_1 \frac{\partial u_1}{\partial x} = -\frac{F(u_2 - u_1) + Q}{c_p T_1} \tag{12}$$

$$k_2 \frac{d_2 p_1}{dt} + k_3 \frac{d_2 p_2}{dt} + \rho_2 \frac{\partial u_2}{\partial x} = 0, \quad \frac{d_1 u_1}{dt} + \frac{1}{\rho_1^\circ} \frac{\partial p_1}{\partial x} = -\frac{F}{\rho_1}$$

$$\frac{d_2 u_2}{dt} + \frac{p_1 - p_2}{\rho_2} \frac{\partial p_1}{\partial x} + \left(\frac{1}{\rho_2^\circ} + \frac{p_1 - p_2}{\rho_2} f_{,2}\right) \frac{\partial p_2}{\partial x} = \frac{E}{\rho_2}$$

$$a_1 = \sqrt{\partial p_1 / \partial \rho_1^\circ}\,|_{s_1}$$

$$k_1 = \alpha_1 / a_1^2 + \rho_1^\circ f_{,1}, \quad k_2 = \alpha_2 g_{,1} - \rho_2^\circ f_{,1}$$

$$k_3 = \alpha_2 g_{,2} - \rho_2^\circ f_{,2}, \quad f_{,i} = \partial f / \partial p_i, \quad g_{,i} = \partial g / \partial p_i, \quad i = 1, 2$$

System (12) can be written in matrix form

$$Z_1 \frac{\partial \mathbf{W}}{\partial t} + Z_2 \frac{\partial \mathbf{W}}{\partial x} = 1, \quad \mathbf{W} = \{p_1, p_2, u_1, u_2\}$$

By using the standard procedure /7/ for finding the characteristic directions $dx/dt = \lambda$ of system (12), we obtain that λ must be an eigenvalue of the linear operator $Z_1^{-1} Z_2$, that is, it must satisfy the following characteristic equation:

$$A(u_1 - \lambda)^2 (u_2 - \lambda)^2 - B(u_1 - \lambda)^2 - C(u_2 - \lambda)^2 + D = 0 \tag{13}$$

$$A = 1 - \frac{\rho_1^\circ f_{,2} k_2}{k_1 k_3}, \quad B = \frac{\rho_2}{k_3}\left(\frac{1}{\rho_2^\circ} + \frac{p_1 - p_2}{\rho_2} f_{,2} - \frac{p_1 - p_2}{k_1 \rho_2} f_{,1} \rho_1^\circ f_{,2}\right)$$

$$C = \frac{\alpha_1}{k_1}, \quad D = \frac{\alpha_1 \rho_2}{k_1 k_3}\left(\frac{1}{\rho_2^\circ} + \frac{p_1 - p_2}{\rho_2} f_{,2}\right)$$

Before proceeding to the investigation of the resulting system in general form, let us consider several special limiting cases.

1°. Let the bulk concentration of the gas, α_1, be solely dependent on p_1, that is, $\partial f / \partial p_2 = 0$, $\partial \varepsilon_{rr} / \partial p_2 = 0$. The relationships (2) then take the form $\varepsilon_{rr} = ap_1$, $\varepsilon_{xx} = ap_2 - bp_1$.

In this case, we find the following characteristic directions:

$$\lambda_{1,2} = u_1 \pm \sqrt{\frac{\alpha_1}{\alpha_1 / a_1^2 + \rho_1^\circ f_{,1}}}, \quad \lambda_{3,4} = u_2 \pm \sqrt{\frac{E}{(1 - \nu^2) \rho_2^\circ}} \tag{14}$$

Hence, in the given special case, system (12) is always hyperbolic. It can be seen from relationships (14) that perturbations will propagate throughout the solid phase with the velocity of longitudinal waves in a thin plate and, throughout the gas phase, at the velocity of the perturbations in "a tube with elastic walls", that is, in a tube with a cross-sectional area which depends on the pressure within it. The relationships for $\lambda_{1,2}$ correspond to the solution obtained in /8, 9/.

2°. Let $\alpha_1 = f_2(p_2)$, that is, let us consider the special case when $p_2 \gg p_1$ so that it may be assumed that $\partial f/\partial p_1 = 0$, $\partial \varepsilon_{rr}/\partial p_1 = 0$. This case corresponds to the model equations of state of a condensed phase (k-phase) $p_2 = \varphi(\alpha_2)$ which are used in /5/. Then, ε_{xx} and ε_{rr} and, consequently, ρ_2 are solely functions of p_2. For example, $\varepsilon_{rr} = -bp_2$, $\varepsilon_{xx} = ap_2$.

In this case, the characteristic directions are:

$$\lambda_{1,2} = u_1 \pm a_1, \quad \lambda_{3,4} = u_2 \pm \sqrt{\frac{E}{(1-\nu^2)\rho_2^\circ}\left(1 + \frac{p_2\nu(1+\nu)}{E}\right)} \tag{15}$$

It is seen that weak perturbations in the gas propagate at the speed of sound a_1 while, in the k-phase, they propagate at a speed which differs from the velocity in plates by the magnitude of a small parameter which depends on p_2. The non-linearity which arises is due to the allowance for the additional dependence of the cross-sectional area of the k-phase on the stress in it: $\alpha_2 = f_2(p_2)$.

3°. Let $\alpha_1 \to 0$. It is clear that, for small α_1, the quantity ε_{rr} is bounded and, in the limit when $\alpha_1 \to 0$, the transverse strain also tends to zero ($\varepsilon_{rr} \to 0$), and the relation between p_1 and p_2 takes the form $p_2 = ap_1/b$. To determine the strain ε_{xx} we have the relationship $\varepsilon_{xx} = (a - b^2/a)\,p_2$.

In this case the system of differential Eqs.(3) is simplified considerably and we obtain four real characteristic directions

$$\lambda_{1,2} = u_1 \pm a_1, \quad \lambda_{3,4} = u_2 \pm \sqrt{\frac{E(1-\nu)}{\rho_2^\circ(1+\nu)(1-2\nu)}} \tag{16}$$

which is indicative of the hyperbolic nature of the system of differential equations. It is seen that, when the bulk concentration of the gas phase is reduced, weak perturbations will propagate in the solid phase at the velocity of the longitudinal waves in an isotropic, unbounded, linearly elastic medium.

4°. Let us consider the case when $\alpha_2 \to 0$. The strains and stresses are connected by relationship (2). The characteristic directions are then determined by formulae which differ from (15) in that p_2 is replaced by $p_2 - p_1$.

Hence, when the bulk content of the k-phase is reduced compared with the gas phase for a fixed overall area of the tube, weak perturbations will propagate throughout the gas at the gas speed of sound in the gas, a_1, while they will propagate throughout the solid at a speed close to the velocity of a longitudinal wave in a thin plate. At a difference in the pressures between the phases of up to 1000 atmospheres, the second term of the radicand for $\lambda_{3,4}$ is two orders of magnitude smaller than the first.

We will now investigate system (12) in the general case when $\alpha_1 = f(p_1, p_2)$ (10) and the link between the stresses and strains is specified by relationship (2). In order to determine the roots of the characteristic Eq.(13), we introduce the notation $u_1 - \lambda = X$, $u_2 - \lambda = Y$, $u_2 - u_1 = \Delta u = Y - X$. For each value of the relative rate of slippage of the phases, the solution of Eq.(13) can be represented as a solution of the system

$$Y = \pm\sqrt{\frac{BX^2 - D}{AX^2 - C}}, \qquad Y = X + \Delta u \tag{17}$$

The real solutions in the X, Y plane correspond to the points of intersection of the line $Y = X + \Delta u$ and a fourth order curve which is symmetrical with respect to the coordinate axes and is specified by the first equation of system (17).

We will now transform the radicand to a more convenient form for investigation. We obtain

$$Y = \pm\sqrt{\frac{c}{a}\,\frac{(X/X_2)^2 - 1}{(X/X_1)^2 - 1}} \tag{18}$$

$$X_1 = \sqrt{b'/(a_0 + a'')}, \quad X_2 = \sqrt{b'/(a_0 + a'/c)}$$

$$a_0 = \frac{\alpha_1}{a_1^2}, \quad a' = \frac{\rho_1^\circ f_{,1}}{\rho_2^\circ}, \quad c = \frac{1}{\rho_2^\circ} + \frac{p_1 - p_2}{\rho_2} f_{,2}$$

$$b' = \alpha_1, \quad a'' = \rho_1^\circ\left(f_{,1} + \frac{b}{a} f_{,2}\right)$$

where a_0, b', a' and a'' are positive coefficients (when $-1 < \nu < 1/2$) and c is of alternating sign.

For a solution of the characteristic equation to exist, it is necessary that the radicand should be non-negative.

Let us now consider the case when $c < 0$, $\Delta p < -1/b$. If, besides this, $a_0 + a'/c > 0$, that is,

$$\Delta p < -\frac{1}{b} - \frac{\rho_1^\circ a\alpha_2 a_1^2}{b\alpha_1} \qquad (19)$$

then the graph of curve (18) has the form shown in Fig.1. It is seen that, for any value of the rate of slippage of the phases, Δu, the line $Y = X + \Delta u$ has two intersections with the curve (18), that is, the characteristic equation has two roots. Hence, the initial system of equations is not hyperbolic for the pressure drops being considered, Δp (19).

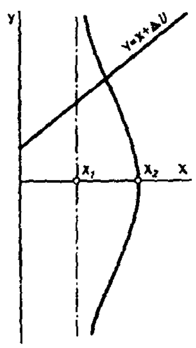

Fig.1

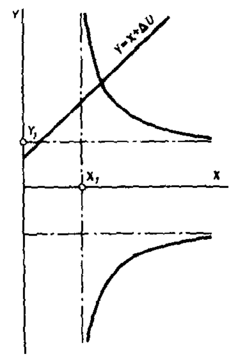

Fig.2

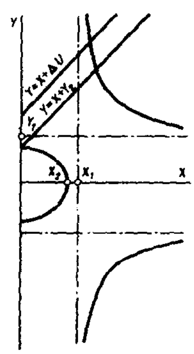

Fig.3

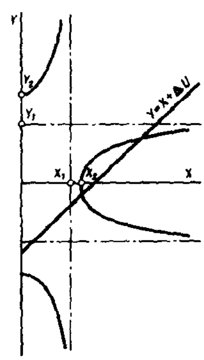

Fig.4

If $a_0 + a'/c < 0$, that is,

$$-\frac{1}{b} - \frac{\rho_1^\circ a\alpha_2 a_1^2}{b\alpha_1} < \Delta p < -\frac{1}{b} \qquad (20)$$

then the graph of curve (18) has the form shown in Fig.2 $(Y_1 = \sqrt{cX_1/(aX_2)})$.

It is seen that, at large values of the difference in the velocities, Δu, and in the case when condition (20) is satisfied, the system of equations becomes hyperbolic. However, this difference in velocities $|\Delta u| = Y - X$ must be greater than

$$\sqrt{\frac{c}{a}} = \sqrt{\frac{1}{a}\left(\frac{1}{\rho_2^\circ} + \frac{p_2 - p_1}{\rho_2^\circ} b\right)}$$

that is, greater than the speed of sound in the k-phase, which is unrealizable in practical problems. Hence, the system is also not hyperbolic in this case for real values of the magnitude of Δu.

When $\Delta p = -1/b$, the curve in Fig.2 is supplemented with the point $X = 0, Y = 0$ and, in the case of the single velocity model $(\Delta u = 0)$, the characteristic equation has four real roots, two of which are multiple.

Let us now consider $\Delta p > -1/b$ $(c > 0)$. If, at the same time, $a'/c > a''$, that is,

$$-\frac{1}{b} < \Delta p < -\frac{1}{b} + \frac{a^2}{(a^2 - b^2)\,b} \qquad (21)$$

then the graphical depiction of the characteristic curve (18) corresponds to that shown in Fig.3 $(Y_2 = \sqrt{c/a})$. The point at the origin of coordinates is transformed into an expanding oval as Δp increases. It is seen that, for real values of the difference between the velocities of the phases which are smaller than the speed of sound in the k-phase.

$$\left(|\Delta u| \lessapprox \sqrt{\frac{c}{a}} = \sqrt{\frac{E}{\rho_2^\circ(1-\nu^2)}\left[1+\frac{\Delta p\nu(1+\nu)}{E}\right]}\right)$$

system (17) has four different real solutions which suggests that the initial system of equations is hyerbolic over the range of variation of Δp being considered. Inequality (21), which expresses the condition of hyerbolicity of the system for real values of the magnitude of the slippage, can be written in the form

$$-\frac{E}{\nu(1+\nu)} < \Delta p < \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{22}$$

Next, when $a'/c < a''$, that is, when

$$\Delta p > -\frac{1}{b} + \frac{a^2}{(a^2-b^2)b} \quad \left(\Delta p > \frac{\nu E}{(1+\nu)(1-2\nu)}\right) \tag{23}$$

the graph of the characteristic curve takes the form shown in Fig.4. At the same time, as the difference between the pressures in the phases, Δp, increases, the curves in Fig.4 become ever more distant from the asymptotes such that, in the case of a small (real) difference between the velocities of the phases, system (12) again becomes non-hyperbolic. For sufficiently large values of the magnitude of the slippage, four points of intersection of the curve (18) with the line $Y = X + \Delta u$, are preserved, that is, the system remains hyperbolic.

In the case when

$$\Delta p = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad \left(\frac{a'}{c} = a''\right)$$

the curves in Fig.4 are converted into two pairs of asymptotes. At the same time, the characteristic equation for any values of Δu has four real roots.

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